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Some examples of non-massive Frobenius manifolds in singularity theory

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Abstract

Let $f, g : \mathbb{C}^2 \to \mathbb{C}$ be two quasi-homogeneous polynomials. We compute the V-filtration of the restriction of f to any plane curve $C_t = g^{-1}(t)$ and show that the Gorenstein generator $dx \wedge dy/dg$ is a primitive form. Using results of A. Douai and C. Sabbah, we conclude that the base space of the miniversal unfolding of $f_t := f|_{C_t}$ is a Frobenius manifold. At the singular fibre C_0 we obtain a non-massive Frobenius manifold.

Keywords: Frobenius manifolds; Gauss–Manin system

1. Introduction

The axiomatic of Frobenius manifolds as originally defined by Dubrovin as for example in [8] represents the geometrisation of the celebrated WDVV or associativity equations in topological quantum field theories (cf. [6]). This geometrisation made it plainly evident that Frobenius manifolds, and hence solutions to WDVV equations, already existed in a very different branch of mathematics, namely singularity theory and more particularly deformations of hypersurface singularities. This work had been carried out by Saito and Saito nearly ten years before (see [17–19]).

The other main source of Frobenius manifolds is quantum cohomology, where the solutions are a priori just formal series and can only be geometrised after some effort if at all. A version of the mirror phenomenon is interpreted in this framework as an isomorphism of two Frobenius manifolds, each coming from one of these two seemingly unrelated sources. In this direction, we have the result of Barannikov [1] establishing an isomorphism between the quantum cohomology of projective spaces and the Frobenius manifold obtained by unfolding the function $x_0 + \cdots + x_n$ on the affine variety $x_0 \cdots x_n = 1$.

As the mirror of $\mathbb{P}^n$ indicates, in order to find potential mirrors of algebraic varieties, it is not enough to look at Frobenius manifolds produced by unfolding of germs of isolated singularities. Global functions on affine varieties are needed. Douai and Sabbah in [7] have adapted the results of Saito to this global affine situation and, under some mild hypotheses, reduced the existence of Frobenius–Saito structures on the base space of the miniversal unfolding to the

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existence of a primitive form for the Gauss–Manin system. They used their results to exhibit Frobenius structures for unfoldings of non-degenerate and convenient Laurent polynomials.

In this article, we construct Frobenius manifolds for unfoldings of quasi-homogeneous functions on quasi-homogeneous plane curves. Let $f, g : \mathbb{C}^2 \to \mathbb{C}$ be quasi-homogeneous polynomials with respect to the same weights. We regard g as a family of plane curves $C_t = g^{-1}(t)$, and consider the restriction $f_t := f|_{C_t}$. We show that the V-filtration of the Gauss–Manin system, and hence the spectral pairs, of f_t can be computed from f_0. In particular, the Gorenstein generator $\alpha := dx \wedge dy/dg$ yields a primitive form with associated spectral number 0. It follows from [7] that the base space of the miniversal deformation of f_t can be endowed with a Frobenius manifold structure. At $t = 0$, the curve C_0 has an isolated singularity. We can use the dualising module ω_{C_0} to define the Gauss–Manin system of f_0 and the Grothendieck residue pairing to construct a non-massive Frobenius manifold.

The motivation behind the construction is the following remark: the unfolding of $f = x^a + y^b$ on $C_t : xy = t$, $t \neq 0$, is the mirror partner of the weighted projective line $\mathbb{P}(a, b)$ (for a and b coprimes, see [14,4]). At $t = 0$, the multiplication and metric in our construction at the origin coincides with the orbifold cohomology of $\mathbb{P}(a, b)$.

2. Preliminaries

Let us recall briefly how to obtain a Frobenius manifold from a meromorphic connection. We closely follow Sabbah (cf. [16]).

Let $G \to B$ be a vector bundle on a manifold B and let the rank of G be equal to the dimension of B, say m. Let F denote the pull-back of G via the projection $\mathbb{P}^1 \times B \to B$. We further assume that F is equipped with a flat meromorphic connection $\widehat{\nabla}$ with a logarithmic pole along $\{0\} \times B$ and a pole of type 1 along $\{\infty\} \times B$.

From these initial data we obtain the following objects:

(i) The residual connection ∇ on $G \to B$: if τ denotes the coordinate on the affine chart $\mathbb{P}^1 \setminus \{\infty\}$ the connection matrix for $\widehat{\nabla}$ is locally written as

$$\Omega^{\widehat{\nabla}} = \Omega_\tau \frac{d\tau}{\tau} + \sum_{i=1}^{m} \Omega_i du_i \tag{1}$$

where Ω_τ and Ω_i are matrices with holomorphic entries. Here $(u_1, \dots, u_m)$ denotes a coordinate system on a neighbourhood in B. The residual connection ∇ on B is given by

$$\Omega^{\nabla} = \sum_{i=1}^{m} \Omega_i(0, u_1, \dots, u_m) du_i.$$

and the integrability of $\widehat{\nabla}$ implies that of ∇.

(ii) The residue endomorphism of $\widehat{\nabla}$, that is, an endomorphism R_0 of $F|_{\{0\}\times B}$ given in local coordinates by $\Omega_\tau(0, u)$. The integrability of $\widehat{\nabla}$ implies that R_0 is covariantly constant with respect to ∇, i.e., $\nabla R_0 = 0$.

(iii) An endomorphism R_∞ of $F|_{\{\infty\}\times B}$, defined (up to a constant) by the choice of a coordinate θ in $\mathbb{P}^1 \setminus \{0\}$. Indeed, the connection at infinity has a pole of type 1. If we use $\theta = \tau^{-1}$ as a coordinate in $\mathbb{P}^1 \setminus \{0\}$ we see from (1) that $\widehat{\nabla}$ is written near ∞ as

$$\frac{1}{\theta}\left(\Omega_\theta \frac{d\theta}{\theta} + \sum_{i=1}^{m} \Omega_i' du_i\right)$$

where $\Omega_\theta = -\theta\Omega_\tau$ and $\Omega_i' = \theta\Omega_i$ and this form Ω_i' has holomorphic entries. The matrix $\Omega_\theta(0, u_1, \dots, u_m)$ defines the endomorphism R_∞ of $F|_B$. The coordinate θ (and hence τ) will be kept fixed throughout this article. Notice that using the canonical isomorphisms $F|_{\{\infty\}\times B} \simeq F|_{\{0\}\times B} \simeq G$ we can think of the above objects as defined on G.

(iv) The Higgs field Φ, defined as follows. We decompose the connection $\widehat{\nabla} = \widehat{\nabla}' + \widehat{\nabla}''$ according to the decomposition of 1-forms $\pi^*_{\mathbb{P}^1\setminus\{0\}}\Omega^1_{\mathbb{P}^1\setminus\{0\}} \oplus \pi^*_B\Omega^1_B$. We write $\widehat{\nabla}'' = d_B + \Omega''$ and set $\Phi = (\theta\Omega'')|_{\theta=0} = \sum_{i=1}^{m} \Omega_i'(0, u)du_i$. It also depends on the choice of the coordinate θ (up to a constant).

The integrability of $\widehat{\nabla}$ implies the following relations between all of the above objects:

$$\begin{aligned} &\nabla^2 = 0, \qquad \nabla R_0 = 0 \\ &\Phi \wedge \Phi = 0, \qquad [R_\infty, \Phi] = 0 \\ &\nabla \Phi = 0, \qquad \nabla R_\infty + \Phi = [\Phi, R_0]. \end{aligned} \tag{2}$$

Let $\mathcal{F}[*(\{0\} \times B)]$ denote the module of sections of F with poles along $\{0\} \times B$ and let $\mathbf{F}$ denote the locally free $\mathcal{O}_B[\theta]$-module $(\pi_B)_*\mathcal{F}[*(\{0\} \times B)]$. We further assume that $\mathbf{F}$ is equipped with a non-degenerate $\mathbb{C}$-linear pairing

$$S : \mathbf{F} \otimes \mathbf{F} \longrightarrow \theta\mathcal{O}_B[\theta]$$

satisfying

$$\begin{aligned} &S(\theta m, m') = \theta S(m, m') = S(m, -\theta m') \\ &\mathrm{Lie}_{\partial_\theta} S(m, m') = S(\widehat{\nabla}_{\partial_\theta} m, m') + S(m, -\widehat{\nabla}_{\partial_\theta} m') \\ &\mathrm{Lie}_{\partial_{u_i}} S(m, m') = S(\widehat{\nabla}_{\partial_{u_i}} m, m') + S(m, \widehat{\nabla}_{\partial_{u_i}} m'). \end{aligned} \tag{3}$$

Expanding S as a series in $\theta = 0$ we get

$$S(m, m') = \theta s^1_\infty(m, m') + \theta^2 s^2_\infty(m, m') + \cdots.$$

It can be checked that s^1_∞ is a non-degenerate, symmetric pairing on $\mathbf{F}/\theta\mathbf{F}$ which is metric with respect to the connection ∇. For a ∇-horizontal section ω of G, we define its associate period mapping $\varphi_\omega : TB \longrightarrow G$ by

$$\varphi_\omega(\xi) := -\Phi(\xi)(\omega).$$

We say that ω as above is

(1) homogeneous if ω is an eigenvector of R_0 and
(2) primitive if φ_ω is an isomorphism.

If ω is a primitive form, we can define a $\mathcal{O}_B$-algebra structure on Θ_B by setting

$$\varphi_\omega(\xi \star \eta) := -\Phi(\xi)\varphi_\omega(\eta). \tag{4}$$

If ω is also homogeneous then the vector field defined by

$$\mathcal{E} := \varphi_\omega^{-1}(R_\infty(\omega)) \tag{5}$$

rescales both the metric and the multiplication, that is,

$$\mathrm{Lie}_{\mathcal{E}}(\star) = \star \quad \text{and} \quad \mathrm{Lie}_{\mathcal{E}}(s^1_\infty) = C \cdot s^1_\infty \tag{6}$$

for some $C \in \mathbb{C}$.

Theorem 1 (*[16]*). *If ω is a primitive form, the triple $(B, \star, s^1_\infty)$ is a Frobenius manifold. If ω is also homogeneous, then the vector field $\mathcal{E}$ defined in (5) is the Euler vector field of the Frobenius manifold $(B, \star, s^1_\infty)$.*

Remark 2. We finish this section with a remark that simplifies enormously the construction of Frobenius manifolds from families of meromorphic connections. Namely, if B is simply connected, it is enough to check the existence of the primitive form at one single value of the parameter space B. This result is proved in a detailed manner in [16], but it goes back to the work of B. Dubrovin on isomonodromic deformations.

3. Functions on curves

Let us recall the definition of the Milnor number of a function f_0 on a curve-germ given by Mond and van Straten in [15].

Definition 3. Let $(C, 0) \hookrightarrow (\mathbb{C}^{n+1}, 0)$ be a reduced curve-germ and let $f_0 : (C, 0) \to (\mathbb{C}, 0)$ be a function non-constant on any branch. The Milnor number μ of f_0 is defined as

$$\mu := \dim_{\mathbb{C}} \frac{\omega_{C,0}}{\mathcal{O}_{C,0} df_0} \tag{7}$$

where $\omega_{C,0} = \mathrm{Ext}^n_{\mathcal{O}_{\mathbb{C}^{n+1},0}}(\mathcal{O}_{C,0}, \Omega^{n+1}_{\mathbb{C}^{n+1},0})$ denotes the dualising module of $\mathcal{O}_{C,0}$.

Remark 4. The authors in [15] show that if the curve is unobstructed (i.e. the second cotangent cohomology group $T^2_{C,0}$ vanishes) then the Milnor number is preserved under flat deformation of $(C, 0)$ and arbitrary deformation of f_0. More precisely, if $g : (\mathcal{C}, 0) \to (B, 0)$ is a flat deformation of $(C, 0)$ and $f : (\mathcal{C}, 0) \to (\mathbb{C}, 0)$ is an extension of f_0 to $(\mathcal{C}, 0)$, then $\omega_{\mathcal{C}/B,0}/\mathcal{O}_{\mathcal{C},0} df$ is a free $\mathcal{O}_{B,0}$-module of rank μ (here $\omega_{\mathcal{C}/B,0}$ denotes the relative version of the dualising module).

In the case of complete intersection curves the Milnor number is relatively easy to compute. If $(C, 0) \subset (\mathbb{C}^{n+1}, 0)$ is a complete intersection curve defined by $g_1, \dots, g_n$, the dualising module $\omega_{C,0}$ can be identified with the module of meromorphic 1-forms ω on $(C, 0)$ such that $\omega \wedge dg_1 \wedge \cdots \wedge dg_n \in \mathcal{O}_{C,0} \otimes \Omega^{n+1}_{\mathbb{C}^{n+1},0}$. It is therefore customary to write $\omega_{C,0} = \mathcal{O}_{C,0}\alpha$ where

$$\alpha = \frac{dx_1 \wedge \cdots \wedge dx_{n+1}}{dg_1 \wedge \cdots \wedge dg_n}. \tag{8}$$

Remark 5. In fact, the dualising module can always be identified with a certain submodule of meromorphic 1-forms with poles at the singular locus of $(C, 0)$, not only in the case of complete intersection curves (e.g. [3]). Interpreting holomorphic 1-forms as meromorphic forms, we obtain the so-called class map $\mathrm{cl} : \Omega_{C,0} \to \omega_{C,0}$. In the case of complete intersections, this map can be described explicitly using α. If M_i denotes the minor of the Jacobian matrix of $g = (g_1, \dots, g_n)$ obtained by deleting the i-th column, we have $dx_i = (-1)^{i-1} M_i \alpha$. This is the class map and we have

$$\mathrm{cl}(\Omega_{C,0}) = \frac{J_g + (g_1, \dots, g_n)}{(g_1, \dots, g_n)} \alpha \tag{9}$$

where J_g denotes the ideal generated by M_i, $i = 1, \dots, n+1$.

Given now $f_0 : (C, 0) \to (\mathbb{C}, 0)$, let f be a representative of f_0 in $\mathcal{O}_{\mathbb{C}^{n+1},0}$. We can write $df_0 = J\alpha$ where J is the Jacobian determinant of the map $\varphi = (f, g_1, \dots, g_n) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$. Hence $\mu = \mathcal{O}_{C,0}/(J)$ and using the Lê–Greuel formula we see that

$$\mu = \mu_1 + \mu_2 \tag{10}$$

where μ_1 denotes the Milnor number of $(C, 0)$ and μ_2 that of the zero-dimensional complete intersection defined by φ.

An unfolding of f over $(B, 0) = (\mathbb{C}^m, 0)$ is a function $F : (\mathbb{C}^{n+1} \times B, 0) \to (\mathbb{C}, 0)$ together with a fibration $(g, \mathbb{I}) : (\mathbb{C}^{n+1} \times B) \to (\mathbb{C}^n \times B, 0)$ such that $F|_{C_0} = f_0$. We say that F is a *miniversal unfolding (resp. versal)* if the Kodaira–Spencer map

$$\Theta_{B,0} \ni \frac{\partial}{\partial u_i} \mapsto \frac{\partial F}{\partial u_i} \in \frac{\mathcal{O}_{C_0 \times B,0}}{(J)} \tag{11}$$

is an isomorphism (resp. a surjection) of $\mathcal{O}_{B,0}$-modules. Here $(u_1, \dots, u_m)$ denote coordinates on $(B, 0)$. Notice that if $\overline{g} : U \to V$ is an appropriate small representative of g, conservation of the Milnor number implies that the map

$$\Theta_V \ni \frac{\partial}{\partial u_i} \mapsto \frac{\partial F}{\partial u_i} \in \overline{g}_* \left(\frac{\mathcal{O}_U}{(J)} \right) \tag{12}$$

is an isomorphism (resp. surjection) of sheaves of $\mathcal{O}_V$-modules. Hence, if C_t denotes the fibre $\overline{g}^{-1}(t)$, the restriction of F to $C_t \times V$ is a miniversal deformation of $F|_{C_t}$ in the usual left-equivalence sense for multigerms.

Notice also that the isomorphisms (11) and (12) induce structures of $\mathcal{O}_B$-algebras on the tangent sheaf Θ_B. It is proved in [5] that these multiplicative structures satisfy a certain integrable condition turning them into F-manifolds (see [9,10]).

3.1. The quasi-homogeneous case

As noted in Remark 4, the Milnor number is locally preserved under deformations. Here we wish to show that in the quasi-homogeneous case it is actually globally preserved. Later, this will justify the use of algebraic forms to study the Gauss–Manin system.

Most of the calculations that follow can be carried out for the case of complete intersection curve singularities and we do so. However, our techniques can only be used to construct Frobenius manifolds for functions on plane curves as it is in this case that we are able to extract information about the spectrum of the restriction of the miniversal unfolding of f_0 to the Milnor fibre of the singularity g.

Let us begin by introducing some notation that will be kept for the remainder of this article. Let $\mathcal{O}$ denote the polynomial ring $\mathbb{C}[x_1, \ldots, x_{n+1}]$. We make $\mathcal{O}$ into a graded ring by assigning the positive rational weight p_i to the variable x_i. Homogeneity will always mean homogeneity with respect to this grading. Let us be given

(1) a polynomial map $g : \mathbb{C}^{n+1} \to \mathbb{C}^n$ where g_i is homogeneous of degree e_i; we denote the fibre over $t \in \mathbb{C}^n$ by C_t and suppose that the 0-fibre C_0 is not smooth (see Remark 6 below);
(2) a homogeneous polynomial $f \in \mathcal{O}$ of degree 1; we write f_t for the restriction of f to the fibre C_t and assume that f_0 is not constant on any branch of C_0.

Remark 6. The smooth case is exceptional as it is the only case for which f belongs to its Jacobian algebra. On the other hand, the smooth case corresponds to the deformation of the A_μ-singularity in one variable, and it is well known that the base space of its miniversal deformation does have a Frobenius structure.

Let $\alpha = dx_1 \wedge \cdots \wedge dx_{n+1}/dg_1 \wedge \cdots \wedge dg_n$ and let ω_g be the relative dualising module. As before, let J be the Jacobian determinant of $(f, g_1, \ldots, g_n)$ so that $df = J\alpha$. As J is also homogeneous, the only critical point of f_0 is the origin and $\mu = \dim_{\mathbb{C}} \mathcal{O}/(g_1, \ldots, g_n, J)$. The following proposition shows that this is also the sum of the Milnor numbers at the critical points of f_t. Let $(t_1, \ldots, t_n)$ be coordinates on the target space of g.

Proposition 7. *The $\mathbb{C}[t_1, \ldots, t_n]$-module $\mathcal{O}/(J)$ is free of rank μ.*

Proof. The module $\mathcal{O}/(J)$ can be seen as a graded module over the graded ring $\mathbb{C}[t_1, \ldots, t_n]$ where t_i acts by multiplication by g_i. As $\omega_{C_0}/\mathcal{O}_{C_0}df \simeq \mathcal{O}/(g_1, \ldots, g_n, J)$ is a finite dimensional vector space it follows from the graded Nakayama lemma that $\mathcal{O}/(J)$ is finitely generated (we recall that the graded version of Nakayama lemma does not require that the module $\mathcal{O}/(J)$ be finitely generated). As $(g_1, \ldots, g_n, J)$ is a regular sequence, the graded version of the Auslander–Buchsbaum formula tells us that $\mathcal{O}/(J)$ is free as $\mathbb{C}[t_1, \ldots, t_n]$-module. □

4. The Gauss–Manin system

We keep the notation and hypothesis introduced in Section 3.1. Let ω_g be the relative dualising module of g. It is a free $\mathcal{O}$-module of rank 1 generated by the form α defined in (8). We define the (algebraic) Gauss–Manin system of f relative to g as the module

$$\mathbf{G} := \frac{\omega_g[\tau, \tau^{-1}]}{(d - \tau df \wedge)\mathcal{O}[\tau, \tau^{-1}]}$$

where d denotes the relative differential with respect to g.

The module $\mathbf{G}$ is a $\mathbb{C}[t_1, \ldots, t_n, \tau, \tau^{-1}]$-module endowed with a partial integrable connection with respect to ∂_τ defined as

$$\widehat{\nabla}_{\partial_\tau}[\omega] = [-f\omega]. \tag{13}$$

We also consider the *(relative) Brieskorn lattice* G, that is, the image of the canonical map $\omega_g[\tau^{-1}] \to \mathbf{G}$. It is a lattice of $\mathbf{G}$ as the following proposition shows:

Proposition 8. *G is a free $\mathbb{C}[t_1, \ldots, t_n, \tau^{-1}]$-module of rank μ.*

Proof. According to Proposition 7, let $h_1, \ldots, h_\mu$ be a basis of the free $\mathbb{C}[t_1, \ldots, t_n]$-module $\mathcal{O}/J$ consisting of homogeneous elements. Let $\omega_i = h_i\alpha$ and let $\omega = a_0\alpha \in \omega_g$. Then there exist unique $c_1, \ldots, c_\mu \in \mathbb{C}[t_1, \ldots, t_n]$ such that $a_0 = c_1h_1 + \cdots + c_\mu h_\mu + a_0'J$, which implies that $\omega = c_1\omega_1 + \cdots + c_\mu\omega_\mu + a_0'df = c_1\omega_1 + \cdots + c_\mu\omega_\mu + \tau^{-1}da_0'$. Writing $da_0' = a_1\alpha$ we see that $\deg a_0 = \deg a_1 + 1$. The proposition follows by iteration. □

Let $\Omega_g = \Omega^1_{\mathbb{C}^n} / \sum_{i=1}^n \mathcal{O}dg_i$ be the module of relative holomorphic (algebraic) forms. We have a relative class map defined analogously to the absolute case (see Remark 5):

$$\mathrm{cl} : \Omega_g \to \omega_g, \qquad \mathrm{cl}(dx_i) = (-1)^{i-1}M_i\alpha \tag{14}$$

We begin by studying the action of ∂_τ on the forms in $\mathrm{cl}(\Omega_g) \subset \omega_g$, i.e., dualising forms without poles. Recall that we are excluding the case in which C_0 is smooth.

Lemma 9. *Let $I = (g_1, \ldots, g_n)$ and let J_g be the ideal generated by all the maximal minors of the Jacobian matrix of g. The sequence*

$$0 \longrightarrow \frac{J_g + I}{I + (J)} \longrightarrow \frac{\mathcal{O}}{I + (J)} \xrightarrow{f\cdot} \frac{\mathcal{O}}{I + (J)} \longrightarrow \frac{\mathcal{O}}{(f) + I + (J)} \longrightarrow 0 \tag{15}$$

is exact.

Proof. Let μ_1 be the Milnor number of C_0 and μ_2 that of the zero-dimensional complete intersection defined by $\varphi := (f, g_1, \ldots, g_n)$. We know that $\mu = \mu_1 + \mu_2$. Since f and g_i are homogeneous, we have

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_{n+1}} \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_{n+1}} \\ \cdots & & \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_{n+1}} \end{pmatrix} \begin{pmatrix} p_1x_1 \\ p_2x_2 \\ \cdots \\ p_{n+1}x_{n+1} \end{pmatrix} = \begin{pmatrix} f \\ e_1g_1 \\ \cdots \\ e_{n+1}g_{n+1} \end{pmatrix}. \tag{16}$$

As J is the determinant of the first matrix on the left, Cramer's rule says

$$J \cdot (p_ix_i) = \begin{array}{c} \begin{array}{ccccc} & & (i) & & \end{array} \\ \begin{vmatrix} \frac{\partial f}{\partial x_1} & \cdots & f & \cdots & \frac{\partial f}{\partial x_{n+1}} \\ \frac{\partial g_1}{\partial x_1} & \cdots & e_1g_1 & \cdots & \frac{\partial g_1}{\partial x_{n+1}} \\ \cdots & & & & \\ \frac{\partial g_n}{\partial x_1} & \cdots & e_{n+1}g_{n+1} & \cdots & \frac{\partial g_n}{\partial x_{n+1}} \end{vmatrix} \end{array} = (-1)^{i+1} f M_i \bmod I \tag{17}$$

where M_i is the minor of the Jacobian matrix of g obtained by deleting the ith column. From here we see that $fJ_g \subset I + (J)$. On the other hand, we have an exact sequence

$$0 \longrightarrow \frac{(f) + I + (J)}{(f) + I} \longrightarrow \frac{\mathcal{O}}{(f) + I} \longrightarrow \frac{\mathcal{O}}{(f) + I + (J)} \longrightarrow 0 \tag{18}$$

where the middle term has dimension $\mu_2 + 1$ (cf. [12], Prop. 5.12). Since the socle of $\mathcal{O}/I + (f)$ is generated by the Jacobian determinant J (e.g. [11]) we conclude that the first term of (18) has dimension 1 and therefore

$$\dim_{\mathbb{C}} \frac{\mathcal{O}}{(f) + I + (J)} = \mu_2. \tag{19}$$

Going back to the original sequence (15) we conclude that the kernel of $f\cdot$ has dimension μ_2. That μ_2 is also the dimension of the first term of (15) follows from one more exact sequence:

$$0 \longrightarrow \frac{J_g + I}{I + (J)} \longrightarrow \frac{\mathcal{O}}{I + (J)} \longrightarrow \frac{\mathcal{O}}{J_g + I} \longrightarrow 0. \tag{20}$$

The middle and last term of the above sequence have dimension $\mu = \mu_1 + \mu_2$ and μ_1 respectively (e.g. [12], Prop. 9.10). Therefore the first term has dimension μ_2 and the lemma follows. □

Using the class map $\mathrm{cl} : \Omega_{C_0} \to \omega_{C_0}$ introduced in Remark 5, the previous lemma can be restated as follows:

Corollary 10. $\frac{\mathrm{cl}(\Omega_{C_0})}{\mathcal{O}_{C_0} df} = \ker\left(f\cdot : \frac{\omega_{C_0}}{\mathcal{O}_{C_0} df} \to \frac{\omega_{C_0}}{\mathcal{O}_{C_0} df} \right)$.

The following notation will be useful for describing the action of ∂_τ on $\mathrm{cl}(\Omega_g)$.

Notation. We set $\mathbf{e} = \sum_{i=1}^{n} e_i$, $\mathbf{p} = \sum_{i=1}^{n+1} p_i$ and for a homogeneous element $h \in \mathcal{O}$, we define

$$\nu(h) := \deg h + \mathbf{p} - \mathbf{e}.$$

We will also write $\nu(\omega) := \nu(h)$ where $\omega = h\alpha$.

Remark 11. Notice that for $\omega = h\alpha$ with h homogeneous we have $\mathrm{Lie}_{\widetilde{E}}(\omega) = \nu(h)\omega$, where $\widetilde{E}$ denotes the Euler vector field on $\mathcal{O}$. Also, if $h \in J_g$ and as before we denote by M_i the minor of the Jacobian matrix of g obtained by deleting the i-th columns, then

$$\deg(h) \geq \min\{\deg M_i : i = 1, \ldots, n+1\} = \mathbf{e} - \mathbf{p} + \min\{p_i : i = 1, \ldots, n+1\}. \tag{21}$$

It follows that $\nu(\omega) > 0$.

Lemma 12. *Let $\omega \in \mathrm{cl}(\Omega_g)$ be a homogeneous 1-form. Then, in* $\mathbf{G}$ *we have*

$$\tau\partial_\tau[\omega] = -\nu(\omega)[\omega] + \sum_{j=1}^{n} t_j \omega_j + \tau \sum_{j=1}^{n} t_j \omega'_j \tag{22}$$

with $\nu(\omega_i) = \nu(\omega) - e_j$ and $\nu(\omega'_j) = \nu(\omega) + 1 - e_j$ (if $\omega_i, \omega'_j \neq 0$).

Proof. By linearity, we can assume that $\omega = h dx_{n+1}$ with h homogeneous. As $dx_{n+1} = (-1)^n M_{n+1}\alpha$, we see that $\nu(\omega) = \deg h + p_{n+1}$. Let us introduce some helpful notation for carrying out the calculation: $i_{\partial_{x_i}}$ denotes the contraction with respect to the vector field ∂_{x_i}, $\mathrm{i}_{n+1} = i_{\partial_n} \circ \cdots \circ i_{\partial_1}$ and $\mathrm{i}_{n+1,j} = i_{\partial_n} \circ \cdots \circ \widehat{i_{\partial_j}} \circ \cdots \circ i_{\partial_1}$. Writing $V = dx_1 \wedge \cdots \wedge dx_{n+1}$ and $dx_{n+1} = \mathrm{i}_{n+1} V$ we have

$$\begin{aligned}
-\tau\widehat{\nabla}_\tau[\omega] &= -\tau\partial_\tau[h\mathrm{i}_{n+1}V] = \tau[h\mathrm{i}_{n+1}(fV)] = \tau[h\mathrm{i}_{n+1}(df \wedge i_{\widetilde{E}}V)] \\
&= \tau\sum_{j=1}^{n}(-1)^{j+1}\left[h(i_{\partial_j}df) \wedge \mathrm{i}_{n+1,j} i_{\widetilde{E}} V\right] + (-1)^n \tau \left[h df \wedge \mathrm{i}_{n+1} i_{\widetilde{E}} V\right] \\
&= \tau\sum_{j=1}^{n}(-1)^{j+1}\left[h(i_{\partial_j}df) \wedge \mathrm{i}_{n+1,j} i_{\widetilde{E}} V\right] + \tau \left[h df \wedge i_{\widetilde{E}} \mathrm{i}_{n+1} V\right] \\
&= [d i_{\widetilde{E}}\omega] + \tau\sum_{j=1}^{n}(-1)^{j+1}\left[h(i_{\partial_j}df) \wedge \mathrm{i}_{n+1,j} i_{\widetilde{E}} V\right] \\
&= [\mathrm{Lie}_{\widetilde{E}}(\omega) - i_{\widetilde{E}} d\omega] + \tau\sum_{j=1}^{n}(-1)^{j+1}\left[h(i_{\partial_j}df) \wedge \mathrm{i}_{n+1,j} i_{\widetilde{E}} V\right] \\
&= \nu(\omega)[\omega] - [i_{\widetilde{E}} d\omega] + \tau\sum_{j=1}^{n}(-1)^{j+1}\left[h(i_{\partial_j}df) \wedge \mathrm{i}_{n+1,j} i_{\widetilde{E}} V\right].
\end{aligned}$$

Multiplying the second summand by $dg_1 \wedge \cdots \wedge dg_n$ and using that $i_{\widetilde{E}} dg_i = e_i g_i$ we see that $i_{\widetilde{E}} d\omega = \sum_{j=1}^{n} t_j \omega_j$ where $\nu(\omega_j) = \nu(\omega) - e_j$ (if $\omega_j \neq 0$), and analogously for the other summand. This can also be seen by noticing that the expression (22) is homogeneous where $\deg \tau = -1$. $\square$

Corollary 13. *For $\omega \in \mathrm{cl}(\Omega_{C_0})$, we have $\tau \partial_\tau [\omega] = -\nu(\omega)[\omega]$ in $\mathbf{G}_0 = \mathbf{G}/\mathfrak{m}_{\mathbb{C}^n,0}\mathbf{G}$.*

4.1. *V-filtration and spectral numbers*

For a fixed point $t \in \mathbb{C}^n$, set $\mathbf{G}_t = \mathbf{G}/\mathfrak{m}_{\mathbb{C}^n,t}\mathbf{G}$ and analogously $G_t = G/\mathfrak{m}_{\mathbb{C}^n,t}G$. Let us recall the definition of the Malgrange–Kashiwara $V_\bullet$-filtration for $\mathbf{G}_t$. It is the unique filtration $V_\bullet(\mathbf{G}_t)$ indexed by $\mathbb{Q}$ such that

(1) $V_\lambda(\mathbf{G}_t)$ is $\mathbb{C}[\tau]$-free and $\mathbb{C}[\tau, \tau^{-1}] \otimes_{\mathbb{C}[\tau]} V_\lambda(\mathbf{G}_t) = \mathbf{G}_t$ for all $\lambda \in \mathbb{Q}$;
(2) $\tau V_\lambda(\mathbf{G}_t) \subset V_{\lambda-1}$, $\partial_\tau V_\lambda(\mathbf{G}_t) \subset V_{\lambda+1}$ and
(3) the action of $\tau\partial_\tau + \lambda$ is nilpotent on the quotient $\mathrm{gr}^V_\lambda(\mathbf{G}_t) := V_\lambda(\mathbf{G}_t)/V_{<\lambda}(\mathbf{G}_t)$.

Such a filtration exists and is unique (e.g. [2], pg. 113). Moreover, there exists a finite subset $A \subset [0, 1)$ such that $\mathrm{gr}^V_\lambda(\mathbf{G}_t) = 0$ for all $\lambda \notin A + \mathbb{Z}$.

The filtration $V_\bullet(\mathbf{G}_t)$ induces a filtration on $G_t/\tau^{-1}G_t$. The corresponding graded part is given by

$$\mathrm{gr}^V_\lambda(G_t)/\tau^{-1}G_t = \frac{V_\lambda(\mathbf{G}_t) \cap G_t}{V_\lambda(\mathbf{G}_t) \cap \tau^{-1}G_t + V_{<\lambda}(\mathbf{G}_t) \cap G_t}.$$

Let $d(\lambda)$ denote the dimension as a complex vector space of $\mathrm{gr}^V_\lambda(G_t)/\tau^{-1}G_t$. The set of pairs $(\lambda, d(\lambda))$ for which $d(\lambda) \neq 0$ is called the *spectrum* of $(\mathbf{G}_t, G_t)$.

We can use Lemmas 9 and 12 to compute the $V_\bullet$-filtration of the Gauss–Manin system of the function f_0 and for the case of plane curves, for any f_t. The linear map $(-f_0)\cdot : \omega_{C_0}/\mathcal{O}_{C_0} df_0 \to \omega_{C_0}/\mathcal{O}_{C_0} df_0$ is nilpotent and homogeneous. Hence its Jordan basis induces a homogeneous basis of G of the following form:

$$\begin{aligned}
&\left[\omega_1^i\right] = \left[(-f)^i \omega_1^0\right], \quad i = 0, \ldots, N_1 \\
&\left[\omega_2^i\right] = \left[(-f)^i \omega_2^0\right], \quad i = 0, \ldots, N_2 \\
&\cdots \\
&\left[\omega_M^i\right] = \left[(-f)^i \omega_M^0\right], \quad i = 0, \ldots, N_M \\
&\left[\omega_{M+1}^0\right], \ldots, \left[\omega_{\mu_2}^0\right].
\end{aligned} \tag{23}$$

Notice that there are exactly μ_2 Jordan blocks (see (15)). It is helpful to set $\nu_i^j = \nu(\omega_i^j)$. Consider now the following change of basis of G:

$$\widetilde{\omega}_i^j = \begin{cases} \left[\omega_i^j\right] + (\nu_i^j - 1)\tau^{-1}\left[\omega_i^{j-1}\right] & \text{if } \nu_i^j > 1 \\ \left[\omega_i^j\right] & \text{if } \nu_i^j \leq 1. \end{cases} \tag{24}$$

Notice that, a priori it could happen that $\nu_i^0 > 1$ and the above definition would not be correct. But this is not the case as the following lemma shows:

Lemma 14. *We have $\nu_i^0 \leq 1$ for all $i = 1, \ldots, \mu_2$.*

Proof. The socle of the zero-dimensional complete intersection defined by $(f) + I$ has degree $1 + \mathbf{e} - \mathbf{p}$. Hence all the elements of degree greater than $1 + \mathbf{e} - \mathbf{p}$ are contained in the image of the multiplication by f and the lemma follows. $\square$

Let us set

$$\lambda_i^j := \begin{cases} 1 & \text{if } \nu_i^j > 1 \\ \nu_i^j & \text{if } 0 \leq \nu_i^j \leq 1 \\ 0 & \text{if } \nu_i^j < 0 \end{cases}$$

and define $\lambda(\tau^k \widetilde{\omega}_i^j) := \lambda_i^j - k$. We extend this definition to an arbitrary element $\omega \in \mathbf{G}_0$ by writing ω as a linear combination of $\widetilde{\omega}_i^j$ with coefficients in $\mathbb{C}[\tau, \tau^{-1}]$ and taking the maximum of $\lambda(\tau^k \widetilde{\omega}_i^j)$ among the terms appearing in the expression of ω. Finally, we set

$$W_\lambda(\mathbf{G}_0) := \{\omega \in \mathbf{G}_0 : \lambda(\omega) \leq \lambda\}. \tag{25}$$

Theorem 15. *The filtration $W_\bullet(\mathbf{G}_0)$ is the Malgrange–Kashiwara $V_\bullet(\mathbf{G}_0)$-filtration. Hence the numbers λ_i^j together with its multiplicities form the spectrum of $(\mathbf{G}_0, G_0)$.*

Proof. The filtration $W_\bullet$ clearly satisfies the first two defining properties of the $V_\bullet$-filtration. It thus suffices to check that $\tau\partial_\tau + \lambda$ is nilpotent on $\mathrm{gr}_\lambda^W \mathbf{G}_0$ for $\lambda \in [0, 1]$. Notice first that by definition we have $f\omega_i^{N_i} \in \mathcal{O}_{C_0} df$. With Corollary 10 it follows that $\omega_i^{N_i} \in \mathrm{cl}(\Omega_{C_0})$. A straightforward calculation together with Corollary 13 shows that

$$\text{if } j < N_i \text{ then } \tau\partial_\tau \widetilde{\omega}_i^j = \begin{cases} \tau\widetilde{\omega}_i^{j+1} & \text{if } \nu_i^j \leq 0 \\ -\nu_i^j \widetilde{\omega}_i^j + \tau\widetilde{\omega}_i^{j+1} & \text{if } 0 < \nu_i^j \leq 1 \\ -\widetilde{\omega}_i^j + \tau\widetilde{\omega}_i^{j+1} & \text{if } \nu_i^j > 1, \end{cases}$$
$$\text{and if } j = N_i \ \tau\partial_\tau \widetilde{\omega}_i^{N_i} = -\lambda_i^{N_i} \widetilde{\omega}_i^{N_i}. \tag{26}$$

We show the nilpotency of $\tau\partial_\tau + \lambda$ with some detail for the first case in (26) as the others are analogous. As $\nu_i^j \leq 0$ we have $j < N_i$ (see Remark 11). If $\nu_i^j < 0$, then $\nu_i^{j+1} < 1$ so that $\tau\partial_\tau \widetilde{\omega}_i^j \in W_{<0}(\mathbf{G})$. If $\nu_i^j = 0$ then $\nu_i^{j+1} = 1$ and we get

$$(\tau\partial_\tau)^2 \widetilde{\omega}_i^j = \tau(\tau\partial_\tau + 1)\widetilde{\omega}_i^{j+1} = \begin{cases} \tau^2 \widetilde{\omega}_i^{j+2} & \text{if } j+1 < N_i \\ 0 & \text{if } j+1 = N_i. \end{cases} \tag{27}$$

In both cases we have $(\tau\partial_\tau)^2 \widetilde{\omega}_i^j \in W_{<0}(\mathbf{G})$. □

Corollary 16. *In the basis of G_0 induced by $\widetilde{\omega}_i^j$, the matrix of the action of ∂_τ takes the form*

$$(A_0 + A_\infty \tau^{-1}) d\tau \tag{28}$$

where A_0 and A_∞ are constant matrices, with A_∞ diagonal. In particular, G_0 extends to a bundle on $\mathbb{P}^1$ with logarithmic connection on $\tau = 0$.

In the case of plane curves, it turns out that the spectrum of the restriction f_t of f to the fibre C_t coincides with that of C_0. More precisely, we define the $W_\bullet(\mathbf{G_t})$-filtration analogously to (25) but using coefficients in $\mathbb{C}[t, \tau, \tau^{-1}]$. In the next lemma we collect two easy remarks that are used repeatedly in the proof of the next theorem.

Lemma 17. *We have $\nu_i^{N_i} \leq 1 - e + p$ for any $i = 0, \ldots, \nu_2$ and*

$$\omega_i^j \in W_{\nu_i^j} \quad \text{if } \nu_i^j \geq 0 \quad \text{and}$$
$$\omega_i^j \in W_0 \quad \text{if } \nu_i^j < 0. \tag{29}$$

Proof. For the first claim notice that the socle of $\mathcal{O}/(g, J)$ has degree $1 + 2e - 2p$ as can be seen from the Hilbert–Poincaré series of this graded ring.

Regarding (29), it clearly holds if $\nu_i^j \leq 1$ for $\omega_i^j = \widetilde{\omega}_i^j \in W_{\nu_i^j}$. If $1 < \nu_i^j \leq 2$, then $j \geq 1$ (see Lemma 14) and

$$\omega_i^j = \widetilde{\omega}_i^j - (\nu_i^j - 1)\tau^{-1}\widetilde{\omega}_i^{j-1} \in W_{\nu_i^{j-1}+1} = W_{\nu_i^j}. \tag{30}$$

If $2 < \nu_i^j \leq 3$ then

$$\omega_i^j = \widetilde{\omega}_i^j - (\nu_i^j - 1)\tau^{-1}\omega_i^{j-1} \in W_{\nu_i^{j-1}+1} = W_{\nu_i^j}$$

for $1 < \nu_i^{j-1} \leq 2$ so (30) applies. The claim follows easily by induction. □

Theorem 18. *If $n = 1$, the filtration $W_\bullet(\mathbf{G}_t)$ is the Malgrange–Kashiwara $V_\bullet(\mathbf{G}_t)$-filtration for any $t \in \mathbb{C}$.*

Proof. The particularity of the family of plane curves is that $(g) = I \subset J_g$. It follows as in the proof of the previous theorem that $\omega_i^{N_i} \in \mathrm{cl}(\Omega_g)$. We only need to check the third defining property of the $V_\bullet$-filtration, the other two being evident. The action of ∂_τ is now

$$\text{if } j < N_i \text{ then } \tau\partial_\tau\widetilde{\omega}_i^j = \begin{cases} \tau\widetilde{\omega}_i^{j+1} & \text{if } \nu_i^j \le 0 \\ -\nu_i^j\widetilde{\omega}_i^j + \tau\widetilde{\omega}_i^{j+1} & \text{if } 0 < \nu_i^j \le 1 \\ -\widetilde{\omega}_i^j + \tau\widetilde{\omega}_i^{j+1} & \text{if } \nu_i^j > 1, \end{cases}$$
$$\text{and if } j = N_i\, \tau\partial_\tau\widetilde{\omega}_i^{N_i} = -\lambda_i^{N_i}\widetilde{\omega}_i^{N_i} + t\omega_{i,1} + \tau t\omega_{i,2}, \tag{31}$$

where we have used the Eq. (22) in Lemma 12. We begin by showing that

$$\omega_{i,1}, \tau\omega_{i,2} \in W_{<\lambda_i^{N_i}}. \tag{32}$$

Let us first consider the case of $\omega_{i,1}$. According to Lemma 12 we have $\nu(\omega_{i,1}) = \nu_i^{N_i} - e$, which in turn implies $\nu(\omega_{i,1}) \le 1 - p$ in view of Lemma 17. If, as in the proof of Proposition 8, we write

$$\omega_{i,1} = \sum_{k,l} c_k^l(t)\omega_k^l + hJ\alpha \tag{33}$$

we see that $h = 0$ for $\nu(hdf) = 1 + \deg(h)$. Also, if $c_k^l(t) \ne 0$, then $\nu(\omega_k^l) \le \nu(\omega_{i,1}) \le 1 - p$ for all the non-zero elements of the sum (33). Notice that then $\nu_k^l < 1$ for the non-zero terms of the sum (33) so that $\widetilde{\omega}_k^l = \omega_k^l$ and (33) becomes

$$\omega_{i,1} = \sum_{k,l} c_k^l(t)\widetilde{\omega}_k^l, \qquad \lambda_k^l \le \nu_i^{N_i} - e \le 1 - p. \tag{34}$$

Since $\omega_i^{N_i} \in \mathrm{cl}(\Omega_g)$, we have $\nu_i^{N_i} > 0$ according to Remark 11. Thus Eq. (34) implies that $\omega_{i,1} \in W_{<\lambda_i^{N_i}}$ no matter whether $\nu_i^{N_i}$ is greater than, less than or equal to 1.

A similar reasoning applies to $\omega_{i,2}$. As $\nu(\omega_{i,2}) = \nu_i^{N_i} - e + 1$, if we write

$$\omega_{i,2} = \sum_{k,l} c_k^l(t)\omega_k^l + hJ\alpha, \tag{35}$$

those ω_k^l occurring in the sum (36) with $c_k^l \ne 0$ must satisfy $\nu_k^l \le \nu(\omega_{i,2}) = \nu_i^{N_i} + 1 - e \le 2 - p$. Also $\deg h = \nu(\omega_{i,2}) - 1 = \nu_i^{N_i} - e$ and as we have the upper bound $\nu_i^{N_i} \le 1 + e - p$, it implies that $\deg h \le 1 - p$. Eq. (35) becomes

$$\omega_{i,2} = \sum_{k,l} c_k^l(t)\omega_k^l + \tau^{-1}dh. \tag{36}$$

where $\nu(h) = 1 - p$. If we write again $dh = \sum d_k^l(t)\omega_k^l + h'J\alpha$, we can reason like in (33) to see that $h' = 0$ and if $d_k^l(t) \ne 0$, then $\nu_k^l \le 1 - p$. We finally obtain that

$$\omega_{i,2} = \sum_{k,l} c_k^l(t)\omega_k^l + \tau^{-1}\sum_{k,l} d_k^l\omega_k^l \tag{37}$$

where $\nu_k^l \le \nu_i^{N_i} + 1 - e \le 2 - p$ for those ω_k^l appearing in the first summand and $\nu_k^l \le \nu_i^{N_i} - e \le 1 - p$ for those in the second one. Using Lemma 17, we see that $\tau\omega_{i,2} \in W_{<\lambda_i^{N_i}}$ no matter whether $\nu_i^{N_i}$ is greater than, equal to or less than 1 (recall again that $\nu_i^{N_i} > 0$).

With (32) in hand, it is possible to prove Theorem 18 by a case-by-case analysis. For example, for the case $j < N_i$ with $\nu_i^j = 0$ Eq. (27) becomes

$$(\tau\partial_\tau)^2\widetilde{\omega}_i^j = \tau(\tau\partial_\tau+1)\widetilde{\omega}_i^{j+1} = \begin{cases}\tau^2\widetilde{\omega}_i^{j+2} & \text{if } j+1 < N_i\\ \tau t\omega_{i,1}+\tau^2 t\omega_{i,2} & \text{if } j+1 = N_i.\end{cases} \tag{38}$$

In the second case above, we have $\nu_i^{N_i} = 1$ and we have seen that $\omega_{i,1}, \tau\omega_{i,2} \in W_{<\lambda_i^{N_i}} = W_{<1}$ so that $(\tau\partial_\tau)^2\omega_i^{N_i-1} \in W_{<0} = W_{<\lambda_i^{N_i-1}}$. $\square$

We can then use the results of [7] to construct Frobenius manifolds on the base space of the miniversal deformation of f_t for $t \neq 0$.

Corollary 19. *If $n = 1$, the class of α in G_t is a primitive form for any t. Hence for any $t \neq 0$, the base space of the miniversal deformation of f_t has the structure of a massive Frobenius manifold.*

Proof. Let $\omega_i^j = h_i^{(j)}\alpha$ be the basis of G_0 defined in (23). Then the unfolding $F = f + \sum_{i=1}^{\mu_2}\sum_{j=0}^{N_i} u_i^{(j)} h_i^{(j)}$ is miniversal. The connection with respect to the deformation parameters is given by

$$\widehat{\nabla}_{\partial_{u_i^{(j)}}}[\omega] = \left[\frac{\partial\omega}{\partial u_i^{(j)}}\right] - \tau\left[\frac{\partial F}{\partial u_i^{(j)}}\omega\right]. \tag{39}$$

As $\alpha = \omega_1^0$ we have

$$\widehat{\nabla}_{\partial_{u_i^{(j)}}}[\alpha] = -\tau[\omega_i^j], \qquad \widehat{\nabla}_{\partial_\tau}[\alpha] = [\omega_1^1] - \sum_{i=1}^{\mu_2}\sum_{j=0}^{N_i} u_i^{(j)}[\omega_i^j], \tag{40}$$

(for the second equation above, notice that $\alpha \notin \mathrm{cl}(\Omega_{C_0})$ since C_0 is singular; thus $N_1 > 0$ and ω_1^1 is defined). It follows that α is a primitive form. The existence of the metric follows from the microlocal Poincaré duality (cf. [7]). Finally, for a generic value of u, all the critical points of F on $C_t \times \{u\}$ are Morse; hence the multiplication is generically semisimple. $\square$

It is known that the metric is given by the sum of the residues at the critical points. More precisely, if $(u_1, \dots, u_\mu)$ are parameters of the base space of the miniversal deformation $F : (\mathbb{C}^2 \times B, 0) \to (\mathbb{C}, 0)$ and $dF = J_F\alpha$ denotes the relative differential, then

$$\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right\rangle_t = \int_{\partial C_t}\left.\left(\frac{\frac{\partial F}{\partial u_i}\frac{\partial F}{\partial u_j}}{J_F}\alpha\right)\right|_{C_t} \tag{41}$$

where ∂C_t is the boundary of an appropriate representative of the Milnor fibre of g.

Corollary 20. *The formula (41) for $t = 0$ together with the multiplication defined by (11) defines the structure of the non-massive Frobenius manifold on B.*

Proof. We have $\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right\rangle_t \to \left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right\rangle_0$ when $t \to 0$. The flatness of $\langle -, -\rangle_t$ implies that of $\langle -, -\rangle_0$ as can be seen, for example, writing out explicit formulas for the curvature in terms of the Christoffel symbols. The existence of a potential can be translated into the flatness of the first structure connection (e.g. [13], Th. 1.5). More precisely, for each t, let ${}^t\nabla$ be the Levi-Civita connection of $\langle -, -\rangle_t$. The first structure connection is defined as

$${}^t\overline{\nabla}_{z,\partial_{u_i}}\partial_{u_j} := {}^t\nabla_{\partial_{u_i}}\partial_{u_j} + z\partial_{u_i} \star_t \partial_{u_j}. \tag{42}$$

It is of course closely related to the Gauss–Manin connection $\widehat{\nabla}$. Notice that $\partial_{u_i} \star_t \partial_{u_j} \to \partial_{u_i} \star_0 \partial_{u_j}$ when $t \to 0$, and hence ${}^t\overline{\nabla} \to {}^0\overline{\nabla}$ and the result follows. $\square$

5. An example: Linear functions on the A_k-singularity

Let us illustrate our construction with a worked-out example. We consider the curve C_0 defined by $g(x, y) = x^k + y^2 = 0$, $k \geq 2$, and the function f_0 given by the restriction of $f(x, y) = x$ to C_0.

Miniversal deformation. The classes of $1, \ldots, x^{k-1}$ form a $\mathbb{C}$-basis of the Jacobian algebra $\mathcal{O}_{C_0}/(2y)$ and hence a miniversal unfolding is given by $F = f + u_1 x^{k-1} + \cdots + u_{k-1}x + u_k$.

Spectrum. For a homogeneous polynomial h we have

$$\nu(h) = \deg(h) - \frac{k-2}{2}.$$

According to Theorem 15, the spectrum of $f_t = f|_{C_t}$ is

$$\begin{aligned} &\left\{\left(0, \frac{k}{2}\right), \left(1, \frac{k}{2}\right)\right\} \quad \text{if } k \text{ is even and,} \\ &\left\{\left(0, \frac{k-1}{2}\right), \left(\frac{1}{2}, 1\right), \left(1, \frac{k-1}{2}\right)\right\} \quad \text{if } k \text{ is odd.} \end{aligned} \tag{43}$$

Nilpotent Frobenius structure. If we set $F' = \frac{\partial F}{\partial x}$, the multiplication table on $\Theta_{B,0}$ is given by the isomorphism

$$\partial_{u_i} \mapsto x^{k-i} \in \pi_* \left(\frac{\mathcal{O}}{(x^k + y^2, 2yF')} \right) \tag{44}$$

where $\mathcal{O}$ denotes the sheaf of holomorphic functions on the variables $x, y, u_1, \ldots, u_k$ and $\pi : C_0 \times (B, 0) \to (B, 0)$ is the canonical projection. The ideal $(x^k + y^2, 2yF')$ defines in $C_0 \times B$ a scheme with two components: $W_1 := \{0\} \times B$ and the (reduced) variety W_2 defined by $F' = 0$. As W_1 already has multiplicity $k = \mu$, the F-manifold structure extends to $B \setminus \pi(W_2)$ (notice that $0 \notin \pi(W_2)$). We see that this F-manifold structure is purely nilpotent, in the sense that if $i \neq k$ (i.e., if ∂_{u_i} is not the identity), we have $\partial_{u_i} \star \cdots \star \partial_{u_i} = 0$ where the product occurs at most k times. We remark that this is always the case if the function f_0 is the restriction of a linear function as all the critical points are provided by the singular curve.

The metric is not easy to give explicitly, only its restriction to $T_0 B$. We have

$$2\pi i \operatorname{Res}(x^j \alpha) = \int_{\partial C_0} \left(\frac{x^j \alpha}{J} \right)\Big|_{C_0} = \int_{\partial C_0} \left(\frac{x^j \mathrm{d}x}{4y^2} \right)\Big|_{C_0} = \int_{\partial C_0} \left(\frac{x^j \mathrm{d}x}{-4x^k} \right)\Big|_{C_0}. \tag{45}$$

Hence the metric in the basis $\partial_{u_i}|_0$ is simply given by the matrix with all its entries equal to $-1/4$ in the anti-diagonal, and 0 everywhere else.

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